Application of the parareal algorithm for acoustic wave propagation

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Plan

Motivation
Parareal algorithm for acoustics
Numerical tests: homogeneous - heterogeneous media
Reducing computational cost: different spatial grids
Summary and current work
Motivation

- Context: simulation of wave propagation in complex 3D media (global, regional or reservoir scales)
  Need of accurate and long-time simulations (hundreds of $\lambda$'s)
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  Need of accurate and **long-time** simulations (hundreds of $\lambda$’s)

- **Methodology**: Spectral Elements Method (SEM)
  - High accuracy and flexibility (FEM) for complex geological media
  - Spectral approximation in space, explicit solver in time (diagonal mass matrix)
  - Efficient *space* parallelization by domain decomposition
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- **Time parallelization?**
  Parareal algorithm presents instabilities for hyperbolic equations
  (Farhat *et al*, 2003, 2006; Bal 2005)
Parareal algorithm for acoustics

\[
\begin{align*}
\frac{\partial v(x, t)}{\partial t} &= c^2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \quad x \in [0, L], \ t > 0 \\
\frac{\partial u(x, t)}{\partial t} &= v(x, t), \quad u(0, t) = u(L, t), \quad u(x, 0) = u_0, \ v(x, 0) = v_0
\end{align*}
\]
Parareal algorithm for acoustics

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Parareal algorithm (Lions, Maday & Turinici (2001)):
\(G\) coarse propagator, \(F\) fine propagator, \(\lambda_n = [u_n, v_n]^T\),

\[
\begin{align*}
\lambda_{n+1} &= G(T_n, T_{n+1}, \lambda_n) \\
\lambda_{n+1} &= F(T_n, T_{n+1}, \lambda_n) - G(T_n, T_{n+1}, \lambda_n)
\end{align*}
\]
Parareal algorithm for acoustics

\[ \frac{\partial v(x, t)}{\partial t} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \quad x \in [0, L], \quad t > 0 \]

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\lambda_{n+1}^{k+1} = G(T_n, T_{n+1}, \lambda_n^{k+1}) + F(T_n, T_{n+1}, \lambda_n^k) - G(T_n, T_{n+1}, \lambda_n^k)
\]

prediction

\[ \lambda_{n+1} = \left\{ \begin{array}{c}
G(T_n, T_{n+1}, \lambda_{n+1}^{k+1}) + F(T_n, T_{n+1}, \lambda_n^k) - G(T_n, T_{n+1}, \lambda_n^k)
\end{array} \right\}
\]

correction
Parareal algorithm for acoustics

\[
\frac{\partial v(x, t)}{\partial t} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \quad x \in [0, L], \ t > 0
\]

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**Parareal algorithm** (Lions, Maday & Turinici (2001)):

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\]

\( \lambda_n^{k+1} \) in parallel, \( \lambda_n^k \) sequential.
Parareal algorithm for acoustics

Time-discontinuous Galerkin: variational formulation *in time*

Hulbert & Hughes (1988, 1992); Li & Wiberg (1995); Kunthong & Thompson (2005)....

\[
\begin{align*}
\text{Find } u, v \in V_h = \hat{y}_h \in S_h (P_p (I_n)) \text{ such that for all } w, \quad \int_{I_n} w u \cdot (M \dot{v} + Ku - F) \, dt + \int_{I_n} w v \cdot (K (\dot{u} - v)) \, dt + w u_n \cdot [u] + w v_n \cdot [v] &= 0
\end{align*}
\]

from Wiberg & Li (1996)
**Parareal algorithm for acoustics**

**Time-discontinuous Galerkin : variational formulation in time**


Find $u, v \in \mathcal{V}_h = \{ y^h \in \bigcup_h (P^p(l_n)) \}$ such that for all $w_u, w_v \in \mathcal{V}_h$,

$$
\int_{l_n} w_v \cdot (M \dot{v} + Ku - F) dt + \int_{l_n} w_u \cdot (K(\dot{u} - v)) dt + w_{u_n} \cdot K[u_n] + w_{v_n} \cdot M[v_n] = 0
$$

where $l_n = (t_n, t_{n+1})$ and $[u_n] = u^+_n - u^-_n$ is the 'jump' at time $t_n$.

Fig. 2. Illustration of the space–time DG finite element method.

from Wiberg & Li (1996)
Numerical test: 1D wave propagation - Ricker source

Spectral Elements Method (GLL points). Periodic boundary conditions

Time advance:
- Newmark for both coarse and fine
- TDG for coarse, Newmark for fine
- TDG for both coarse and fine

Physical parameters:

\[ v = 2000 \, \text{m/s} \]
\[ f_0 = 2.5 \, \text{Hz} \]
\[ x_0 = 2500 \, \text{m} \]
Numerical test: 1D wave propagation - Ricker source

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Physical parameters:
\[ v = 2000 \text{ m/s} \]
\[ f_0 = 2.5 \text{ Hz} \]
\[ x_0 = 2500 \text{ m} \]

Parareal parameters:
Total simul time \( T = 5 \text{ s} \)
50 time slices \( \Delta T = 0.1 \text{ s} \)
Coarse solver \( DT = 4.e-4 \text{ s} \)
Fine solver \( dt = 2.e-5 \text{ s} \)

Mesh parameters:
50 elements, 6 GLL \( \rightarrow \) \( \text{CFL}_{Coa} = 0.68 \), \( \text{CFL}_{Fin} = 0.034 \)
Numerical test: 1D wave propagation - Ricker source

Explicit Newmark for both coarse and fine solvers

![Graph showing error vs. time slice number]
Numerical test: 1D wave propagation - Ricker source

TDG for coarse solver, Explicit Newmark for fine solver
Numerical test: 1D wave propagation - Ricker source

TDG for both coarse and fine solvers

[Graph showing error vs. time slice number for different iterations]

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Parareal algorithm for acoustic wave propagation
Parallel Speed-Up

\[
S_p = \frac{T}{(k + 1) \frac{T}{Dt} + k \frac{\Delta T}{dt}} = \frac{1}{(k + 1) \frac{dt}{Dt} + k \frac{\Delta T}{T}},
\]

using \( P = \frac{T}{\Delta T} \),

\[
S_p = \frac{P}{(k + 1) P \frac{dt}{Dt} + k}
\]
Parallel Speed-Up

\[ S_p = \frac{T}{dt} \left( k + 1 \right) + k \frac{\Delta T}{dt} = \frac{1}{(k + 1)\frac{dt}{Dt} + k \frac{\Delta T}{T}} , \]

using \[ P = \frac{T}{\Delta T} , \]

\[ S_p = \frac{P}{(k + 1)P\frac{dt}{Dt} + k} \]

In the previous case, \[ Dt/dt = 20, \, P = 50, \, k = 2 \quad \rightarrow \quad S_p = 5.3 \]
Parallel Speed-Up

\[ S_p = \frac{\frac{T}{dt}}{(k + 1)\frac{T}{Dt} + k\frac{\Delta T}{dt}} = \frac{1}{(k + 1)\frac{dt}{Dt} + k\frac{\Delta T}{T}} \]

using \( P = \frac{T}{\Delta T} \),

\[ S_p = \frac{P}{(k + 1)P\left(\alpha\frac{dt}{Dt}\right) + k} \]

In the previous case, \( Dt/dt = 20, P = 50, k = 2 \) \( \rightarrow S_p = 5.3 \)
In order to reduce the relative \textit{per-step} cost ($\alpha$) of $F_{\Delta T}$ and $G_{\Delta T}$, two strategies are proposed:
In order to reduce the relative per-step cost ($\alpha$) of $F_{\Delta T}$ and $G_{\Delta T}$, two strategies are proposed:

1. Simplified physical model for the coarse solver
Numerical test : 1D wave propagation - Ricker source

In order to reduce the relative per-step cost \((\alpha)\) of \(\mathcal{F}_{\Delta T}\) and \(\mathcal{G}_{\Delta T}\),

Two strategies are proposed:

1. Simplified physical model for the coarse solver
2. Coarser spatial grid resolution in the coarse solver
Reducing coarse simulation cost: 1) simpler physics

Acoustic homogeneization (Capdeville, Guillot & Marigo, 2009)

At order 0, we solve

$$\rho^* \partial_{tt} u^0 - \partial_x \sigma^0 = f$$

$$\sigma^0 = E^* \partial_x u^0$$

with

$$E^*(x) = \left( F_{k_0} \left( \frac{1}{E} \right) \right)^{-1}(x)$$

$$\rho^*(x) = F_{k_0}^*(\rho)(x)$$

where $F_{k_0}$ is a low-pass filter defined by a cut-off wavenumber $k_0$. 
Reducing coarse simulation cost: 1) simpler physics

Fine: Heterogeneous medium  Coarse: homogeneized medium

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Parareal algorithm for acoustic wave propagation
Reducing coarse simulation cost: 1) simpler physics

Fine: Heterogeneous medium

Coarse: homogeneized medium

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Parareal algorithm for acoustic wave propagation
Reducing coarse simulation cost: 1) simpler physics

Fine: Heterogeneous medium

Coarse: homogeneous medium
Reducing coarse simulation cost: 1) simpler physics

Fine: Heterogeneous medium

Coarse: Homogeneous medium

Parareal algorithm for acoustic wave propagation

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Parareal – Sequential

iteration 0

iteration 1
Reducing coarse simulation cost: 1) simpler physics

Fine: Heterogeneous medium
Coarse: homogeneous medium

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Parareal algorithm for acoustic wave propagation
Reducing coarse simulation cost: 1) simpler physics

Fine: Heterogeneous medium  Coarse: homogeneous medium

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Parareal algorithm for acoustic wave propagation
Reducing coarse simulation cost: 1) simpler physics

Fine: Heterogeneous medium

Coarse: homogeneous medium

Homogenous medium for coarse solver is not enough!!
Reducing coarse simulation cost: 2) different spatial grids

\[ \lambda_{n+1}^{k+1} = \Pi^N_M G(\Pi^M_N \lambda_n^{k+1}) + \mathcal{F}(\lambda_n^k) - \Pi^N_M G(\Pi^M_N \lambda_n^k) \]

where \( \Pi^N_M \) is the L2-prolongation operator from \( \mathcal{P}^M \) to \( \mathcal{P}^N \) \((M < N)\), \( \Pi^M_N \) is the L2-restriction operator from \( \mathcal{P}^N \) to \( \mathcal{P}^M \).
Reducing coarse simulation cost: 2) different spatial grids

\[ \lambda_{n+1}^{k+1} = \Pi_M^N G(\Pi_N^M \lambda_n^{k+1}) + F(\lambda_n^k) - \Pi_M^N G(\Pi_N^M \lambda_n^k) \]

where

- \( \Pi_M^N \) is the L2-prolongation operator from \( P^M \) to \( P^N \) (\( M < N \)),
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\[ \int_{\Lambda} (u^c - u^f) \phi_j \, dx = 0 \quad \text{for all} \quad \phi_j \in P^{M-2}, u^c \in P^M, u^f \in P^N \]

\[ \int_{\Lambda} u^c \phi_j \, dx = \int_{\Lambda} u^f \phi_j \, dx \]
Reducing coarse simulation cost: 2) different spatial grids

$$\lambda_{n+1}^{k+1} = \nabla_{M}^{N} G(\nabla_{N}^{M} \lambda_{n}^{k+1}) + F(\lambda_{n}^{k}) - \nabla_{M}^{N} G(\nabla_{N}^{M} \lambda_{n}^{k})$$

where $\nabla_{M}^{N}$ is the L2-prolongation operator from $\mathcal{P}^{M}$ to $\mathcal{P}^{N}$ ($M < N$), $\nabla_{N}^{M}$ is the L2-restriction operator from $\mathcal{P}^{N}$ to $\mathcal{P}^{M}$.

$$\int_{\Lambda} (\mathbf{u}^{c} - \mathbf{u}^{f}) \phi_{j} \, dx = 0 \quad \text{for all} \quad \phi_{j} \in \mathcal{P}^{M-2}, \mathbf{u}^{c} \in \mathcal{P}^{M}, \mathbf{u}^{f} \in \mathcal{P}^{N}$$

$$\int_{\Lambda} \mathbf{u}^{c} \phi_{j} \, dx = \int_{\Lambda} \mathbf{u}^{f} \phi_{j} \, dx$$

using GLL quadrature rules ($\{\eta_{k}\}_{0}^{M}$ for l.h.s. and $\{\xi_{k}\}_{0}^{N}$ for r.h.s.),

$$\sum_{k=0}^{M} \rho_{k}^{c} \phi_{j}(\eta_{k}) \mathbf{u}_{k}^{c} = \sum_{k=0}^{N} \rho_{k}^{f} \phi_{j}(\xi_{k}) \mathbf{u}_{k}^{f}.$$
Reducing coarse simulation cost : 2) different spatial grids

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$$\int_{\Lambda} (u^c - u^f) \phi_j \, dx = 0 \quad \text{for all} \quad \phi_j \in \mathcal{P}^{M-2}, u^c \in \mathcal{P}^M, u^f \in \mathcal{P}^N$$

$$\int_{\Lambda} u^c \phi_j \, dx = \int_{\Lambda} u^f \phi_j \, dx$$

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$$\sum_{k=0}^{M} \rho_f^c \phi_j(\eta_k) u^c_k = \sum_{k=0}^{N} \rho_f^f \phi_j(\xi_k) u^f_k.$$
Reducing coarse simulation cost: 2) different spatial grids

\[ \lambda_{n+1}^{k+1} = \Pi_M^N G(\Pi_N^M \lambda_n^{k+1}) + \mathcal{F}(\lambda_n^k) - \Pi_M^N G(\Pi_N^M \lambda_n^k) \]

where \( \Pi_M^N \) is the L2-prolongation operator from \( P^M \) to \( P^N \) (\( M < N \)), \( \Pi_N^M \) is the L2-restriction operator from \( P^N \) to \( P^M \).

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\[ \sum_{k=0}^{M} \rho_k \phi_j(\eta_k) u^c_k = \sum_{k=0}^{N} \rho_k \phi_j(\xi_k) u^f_k. \]

using \( \phi_j(\eta_k) = \delta_{jk}, \; k = 1, \ldots, M - 1 \), and the edge constraints \( u^f_0 = u^c_0 \) and \( u^f_N = u^c_M \), we obtain,

\[ M_c u_c = R u_f, \quad \Rightarrow \quad u_c = \Pi_N^M u_f, \quad \text{with} \quad \Pi_N^M = M_c^{-1} R. \]
Reducing coarse simulation cost: 2) different spatial grids

\[
\lambda_{n+1}^{k+1} = \Pi_M^N \mathcal{G}(\Pi_N^M \lambda_n^{k+1}) + \mathcal{F}(\lambda_n^k) - \Pi_M^N \mathcal{G}(\Pi_N^M \lambda_n^k)
\]

where \(\Pi_M^N\) is the L2-prolongation operator from \(\mathcal{P}^M\) to \(\mathcal{P}^N\) \((M < N)\), \(\Pi_N^M\) is the L2-restriction operator from \(\mathcal{P}^N\) to \(\mathcal{P}^M\).
Reducing coarse simulation cost: 2) different spatial grids

\[ \lambda_{n+1}^{k+1} = \Pi_{NM}^N G(\Pi_{NM}^M \lambda_n^{k+1}) + \mathcal{F}(\lambda_n^k) - \Pi_{NM}^N G(\Pi_{NM}^M \lambda_n^k) \]

where \( \Pi_{NM}^N \) is the L2-prolongation operator from \( P^N \) to \( P^M \) (\( M < N \)), \( \Pi_{NM}^M \) is the L2-restriction operator from \( P^N \) to \( P^M \).

\[ \int_{\Lambda} (u^f - u^c) \phi_j \, dx = 0 \quad \text{for all} \quad \phi_j \in P^{N-2}, u^c \in P^M, u^f \in P^N \]

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\[ \int_{\Lambda} u^f \phi_j \, dx = \int_{\Lambda} u^c \phi_j \, dx \]

using fine GLL quadrature (\( \{\xi_k\}_0^N \) for both l.h.s. and r.h.s.),

\[ \sum_{k=0}^{N} \rho_k^f \phi_j(\xi_k) u_k^f = \sum_{i=0}^{M} \sum_{k=0}^{N} \rho_k^f \phi_j(\xi_k) h_i(\xi_k) u_i^c. \]
Reducing coarse simulation cost: 2) different spatial grids

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\lambda_{n+1}^{k+1} = \Pi_N^M G(\Pi_N^M \lambda_n^k) + F(\lambda_n^k) - \Pi_N^M G(\Pi_N^M \lambda_n^k)
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\int_{\Lambda} (u^f - u^c) \phi_j \, dx = 0 \quad \text{for all } \phi_j \in \mathcal{P}^{N-2}, u^c \in \mathcal{P}^M, u^f \in \mathcal{P}^N
\]

\[
\int_{\Lambda} u^f \, \phi_j \, dx = \int_{\Lambda} u^c \, \phi_j \, dx
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using fine GLL quadrature (\(\{\xi_k\}_0^N\) for both l.h.s. and r.h.s.),

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\sum_{k=0}^N \rho_k^f \phi_j(\xi_k) u_k^f = \sum_{i=0}^M \sum_{k=0}^N \rho_k^f \phi_j(\xi_k) h_i(\xi_k) u_i^c.
\]

using \(\phi_j(\xi_k) = \delta_{jk}, k = 1, \ldots, N - 1\), and the edge constraints \(u_0^f = u_0^c\) and \(u_N^f = u_N^c\), we obtain,
Reducing coarse simulation cost: 2) different spatial grids

\[
\lambda_{n+1}^{k+1} = \Pi_N^M G(\Pi_N^M \lambda_n^{k+1}) + \mathcal{F}(\lambda_n^k) - \Pi_N^M G(\Pi_N^M \lambda_n^k)
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\int_\Lambda (u^f - u^c) \phi_j \, dx = 0 \quad \text{for all} \quad \phi_j \in \mathcal{P}^{N-2}, \ u^c \in \mathcal{P}^M, \ u^f \in \mathcal{P}^N
\]

\[
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using fine GLL quadrature \(\{\xi_k\}_0^N\) for both l.h.s. and r.h.s.,

\[
\sum_{k=0}^{N} \rho_k^f \phi_j(\xi_k) \ u_k^f = \sum_{i=0}^{M} \sum_{k=0}^{N} \rho_k^f \phi_j(\xi_k) h_i(\xi_k) \ u_i^c.
\]

using \(\phi_j(\xi_k) = \delta_{jk}, \ k = 1, \ldots, N - 1\), and the edge constraints \(u_0^f = u_0^c\) and \(u_N^f = u_M^c\), we obtain,

\[
M_f u_f = M_f P u_c, \quad \Rightarrow \quad \Pi_N^M = P, \quad \text{with} \quad (P)_{ki} = h_i(\xi_k) \quad \Rightarrow \quad \text{Interpolation operator}
\]
Reducing coarse simulation cost: 2) different spatial grids

Prolongation

Restriction
Reducing coarse simulation cost: 2) different spatial grids

Eigenvalue decomposition

\[ (-\omega_j^2 M + K) u_j = 0, \quad j = 1, \ldots, N_{dof} \]

Expansion of ricker wavelet in \( \{u_j\}_{j=1}^{N_{dof}} \):

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Parareal algorithm for acoustic wave propagation
Reducing coarse simulation cost: 2) different spatial grids

Eigenvalue decomposition

\[ (-\omega_j^2 M + K) \mathbf{u}_j = 0, \quad j = 1, \ldots, N_{dof} \]

Expansion of ricker wavelet in \( \{ \mathbf{u}_j \}_{j=1}^{N_{dof}} \):

Excitation of spurious modes!!
Reducing coarse simulation cost: 2) different spatial grids

Eigenvalue decomposition

\[ (-\omega_j^2 M + K) u_j = 0, \quad j = 1, \ldots, N_{dof} \]

Expansion of ricker wavelet in \( \{u_j\}^{N_{dof}}_{j=1} \):
Reducing coarse simulation cost: 2) different spatial grids

Ricker wavelet in homogeneous medium (c = 2000 m/s)

Coarse solver: 50 elements - 6 GLL
Fine solver: 50 elements - 9 GLL
Reducing coarse simulation cost: 2) different spatial grids

Error at $t = 5$ sec

Spectra at $t = 5$ sec
Reducing coarse simulation cost: 2) different spatial grids
Reducing coarse simulation cost: 2) different spatial grids

Application of a low-pass filter \((f_c = 10 \text{ Hz}, \lambda_c = 200 \text{ m})\) after each Prolongation-Restriction → convergence to the fine solution
Reducing coarse simulation cost: 2) different spatial grids

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Parareal algorithm for acoustic wave propagation
Reducing coarse simulation cost: 2) different spatial grids

![Graph of Error at t = 5 sec]

![Graph of Spectra at t = 5 sec]

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Parareal algorithm for acoustic wave propagation
Reducing coarse simulation cost: 2) different spatial grids

It is NOT an easy task to define the filter cut-off frequency (...even worse in heterogeneous media!!)

\[ f_c = 8 \text{ Hz (} \lambda_c = 250 \text{ m)} \]

\[ f_c = 10 \text{ Hz (} \lambda_c = 200 \text{ m)} \]

\[ f_c = 20 \text{ Hz (} \lambda_c = 100 \text{ m)} \]

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Parareal algorithm for acoustic wave propagation
Instabilities seen in the parareal algorithm for the acoustic wave equation discretized by SEM + Newmark solvers can be mitigated by the use of **Time-Discontinuous Galerkin method**
Summary and current work

- Instabilities seen in the parareal algorithm for the acoustic wave equation discretized by SEM + Newmark solvers can be mitigated by the use of Time-Discontinuous Galerkin method.

- Satisfactory convergence properties in highly heterogeneous media → coarse solver in a homogeneized medium.

Next step: can we define Prolongation-Restriction operators that do not excite spurious frequencies?
Summary and current work

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Fine: Heterogeneous medium  Coarse: homogenized medium

explicit Newmark time-scheme
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Parareal algorithm for acoustic wave propagation

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Coarse: Homogenized medium

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Numerical test: 1D wave propagation - Ricker source

TDG for both coarse and fine solvers.

50 seconds simulation, $Dt/dt = 20$, $\Delta T = 1\, s$