Dependence of rock properties on the Lode angle: Experimental data, constitutive model, and bifurcation analysis

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A B S T R A C T

The overwhelming majority of experimental tests on rocks have only been conducted for a single value of the Lode angle \( \theta \) corresponding to the axisymmetric compression (AC). There are now sufficiently extensive data sets from both AC and axisymmetric extension (AE) tests (corresponding to two extreme \( \theta \) values) for two materials (synthetic rock analog GRAM1 and Solnhofen Limestone). These data cover a wide range of the confining pressure (from brittle faulting to ductile flow). Very recently the data from true 3-D tests (for different \( \theta \)) also covering both brittle and ductile fields were published for Castlegate and Bentheim Sandstone as well. The results from all these tests summarized and processed in this paper constitute a solid basis which allows general conclusions to be drawn about the dependence of rock behavior on \( \theta \). In all cases, the yield/failure envelopes were shown to be \( \theta \)-dependent so that the material strength at low mean stress \( \sigma \) is smaller under AE than under AC, while at high \( \sigma \), it is the opposite. The brittle-ductile transition under AE occurs at \( \sigma/C_24 \) 1.5 times greater than under AC, meaning that under AE the material is more prone to fracture development. The angle between the most compressive stress and the forming deformation localization bands is systematically higher for AE than for AC for the same \( \sigma \). Based on these data we formulate a new three-invariant constitutive model with convex and concave yield functions (YFs) which is used for the bifurcation analysis. The results of this analysis agree with the experimental data (for both YFs) and reveal that the \( \theta \)-dependence of rock properties encourages the strain localization. The major factors defining this dependence are the \( \theta \)-dependence of the YFs but also of the dilatancy factor which is greater for AE than for AC. The theoretical results show that the failure (deformation band) plane can deviate from the intermediate stress direction and can become parallel to the maximum compressive stress at high \( \sigma \) for the concave YF.

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1. Introduction

It has long been known that the strength of geomaterials under compression \( \tau_{pk} \) can be considerably higher than that under extension \( \tau_{pk} \) for the same mean stress \( \sigma \) (Mogi, 1967, 1971; Chang and Haimson, 2000; Haimson and Chang, 2000; Haimson and Rudnicki, 2010; Haimson, 2011; Lee and Haimson, 2011), where \( \tau \) is the von Mises stress, the superscript “pk” indicates...
and ψ functions defined in Eq.(3).

unit normal to deformation localization bands

total, elastic, and inelastic strain tensors,

ψ 2 τ equal to

maximum compression stress

dilatancy factor

for Bentheim and Castlegate sandstones

on synthetic Granular Rock Analog Material (GRAM1) consisting of

critical hardening modulus for the Drucker-Prager model, defined in Eqs.(23)

bulk modulus

direction in Eq.(10).

stress tensor (internal friction coefficient

mean stress at the crest of initial yield envelopes

third invariant of stress deviator

ψ 1 angle between σ2-paralleled deformation localization bands (planes) and σ1 direction

ψ 2 angle between σ1-paralleled deformation bands and σ2 direction

Δψ ψ(θ = 0°) − ψ(θ = 60°)  

Δψ B, Δψ C, Δψ for Bentheim and Castlegate sandstones

Δψ 1, Δψ 2, Δψ for GRAM1 material, and Solnhofen limestone

ξ equal to cos 2ψ, Eq. (32).

n1 unit normal to deformation localization bands in the principal stress space

AC axisymmetric compression

AE axisymmetric extension

YS yield surface

YF yield function

σ bdt mean stress at brittle-ductile transition

σ bdt , σ bdt c mean stress at brittle-ductile transition for AC and AE, respectively

ψ bdt , ψ bdt c at brittle-ductile transition for AC and AE, respectively

ψ von Mises stress

F yield function

Φ plastic potential function

τ (σ), τ ex(σ) initial yield functions for AC and AE, respectively

σ0 mean stress at the intersection of τc(σ), τ ex(σ) 

τ pk, τ pk ex von Mises stress at stress peaks for AC and AE loading, respectively

P c confining pressure in conventional tests

P * confining pressure in hydrostatic tests at the onset of grain crushing

q τ coefficients in Eq. (1); (τ = 1, 2...5)

a m coefficients in the function σ1(σ2, σ3) given in the caption of Fig. 2; (m = 1, 2...6)

A, B, C functions defined in Eq. (3).

w, w b, w 2 coefficients (exponents) in the yield function, Eq. (2).

α internal friction coefficient

β dilatancy factor

f j equal to ∂F/∂σ j defines the internal friction coefficient α

f j equal to ∂F/∂σ j defines the internal friction coefficient

f j equal to ∂F/∂σ j defines the internal friction factor β

e ij, e ij p, e ij t total, elastic, and inelastic strain tensors, respectively

e bdt p inelastic strain deviator tensor

y p accumulated inelastic equivalent shear strain

a 0 parameter linking α and β in Eq. (10);
dλ non-negative scalar function in the flow rule, Eq. (8).

H hardening modulus

h = H/G normalized hardening modulus

h cr critical hardening modulus h when deformation bands are parallel to σ1, Eq. (38).

h cr * critical hardening modulus when deformation bands are parallel to σ2, Eq. (39).

h cr ** critical hardening modulus for the Drucker-Prager model, Eq. (41).

Δh cr equal to h cr − h cr *

Δh cr ** equal to h cr − h cr **

Q and R defined in Eqs. (31), (35), (36).

Lijkl, L bdt, L bdt ijk total, elastic, and inelastic stiffness tensors (i, j, k, l = 1, 2, 3)

h, l i, l i, 0 defined in Eqs. (23)–(28).

stands for the peak stress values corresponding to the onset of the material rupture, and the subscripts “c” and “ex” are compression and extension, respectively. Using an exceptionally large data set for the low porosity Solnhofen limestone from axisymmetric compression (AC) and axisymmetric extension (AE) conventional tests conducted under different confining pressure P c, Heard (1960) was the first to show that this is true only up to a certain value of σ. Above this value the relation between τ pk and τ pk c is inverted, τ pk c becoming greater than τ pk. This author also demonstrated for the first time that the transition from brittle to ductile behavior under extension occurs at σ value, σ c ex, almost twice (~1.7) that under compression, σ c bdt (the superscript “bdt” stands for brittle-ductile transition). In other words, rock behavior under extension is much more brittle than under compression. Therefore the extension loading is more prone to fracture development even at high pressure. These fundamental discoveries did not receive much attention from the geomechanics community and until recently were not confirmed because the overwhelming majority of rock tests were limited to a single AC loading type. Heard’s results were only confirmed 50 years later by Nguyen et al. (2011). These authors conducted a wide series of both AC and AE tests under various P c on synthetic Granular Rock Analog Material (GRAM1) consisting of “welded” TiO2 particles. The nature of GRAM1 is obviously very different from that of Solnhofen limestone and it has more than two orders of magnitude lower strength, but the mechanical behavior of these two materials is very similar including the σ c ex /σ c bdt ratio,
which is \( \sim 1.5 \) for GRAM1 (as against 1.7 for Solnhofen limestone).

Fig. 1 shows failure/yield envelopes \( \varepsilon(\sigma) \) for these two materials, where \( \varepsilon \) and \( \sigma \) are plotted at the stress peaks which exist only in the brittle deformation (faulting) regime, i.e., at \( \sigma < \sigma^{\text{brit}} \). At higher \( \sigma \), the inelastic deformation of the material causes it to continuously harden and \( \varepsilon \) to increase. The onset of this deformation (corresponding to the initial yield envelope) is defined by the experimenters either by comparing results of hydrostatic and deviatoric tests (e.g., Wong et al., 1997), or directly by separation of the total measured strain on elastic and inelastic components (Mas and Chemenda, 2014, 2015). From these studies it is known that the continuation of the initial failure/yield envelope into the ductile domain for AC conditions is typically as shown in Fig. 1 by black dashed lines. This envelope intersects the hydrostatic axis at \( \sigma = P^* \) (Fig. 1), which corresponds to the grain crushing pressure at hydrostatic loading. For AE (as well for true 3-D loading), the envelopes were not experimentally constrained for high \( \sigma \), but it is logical to assume that at least for the isotropic materials, both curves (for AC and AE) should meet at the hydrostatic axes. This gives the general shape of the initial yield envelope for the AE loading as shown in Fig. 1 by red lines. The envelopes for AC and AE clearly do not coincide.

It should be noted that at \( \sigma < \sigma^{\text{brit}} \) the initial yield envelope is considered above to coincide with the failure envelope as is usually assumed (e.g., Wong et al., 1997). This is true, however, only as a first approximation because the inelastic strain before the failure in this domain of brittle faulting, although small, is not zero and increases with \( \sigma \) (Mas and Chemenda, 2015).

Closely related to the deformation regime (brittle versus ductile) is the angle \( \psi \) between the most compressive stress \( \sigma_1 \) and the rupture (fault) or the deformation localization band plane forming in the deforming specimen. This angle is well known from AC tests to increase with \( \varepsilon \) from \( \sim 0^\circ \) to \( \sim 35^\circ \) to \( \sim 50^\circ \) around the brittle-ductile transition and then to \( \sim 90^\circ \) at high \( \sigma \), which corresponds to the pure compaction deformation bands (the whole \( \sigma \) range considered in this paper is between zero and \( P^* \)). Under AE, \( \psi \) is systematically higher than for AC for the same \( \sigma \) (Heard, 1960; Handin et al., 1967; Bésuelle et al., 2000; Nguyen et al., 2011).

Along with the conventional axisymmetric tests where two principal stresses are equal (for this reason these tests are also called false 3-D tests), a limited number of true 3-D tests have been conducted as well since Mogi (1967). These tests were typically restricted to relatively small \( \sigma \) values corresponding to the brittle deformation regime for which \( \bar{\tau}^{\text{pk}}>\bar{\tau}^{\text{ext}} \) (Mogi, 1967, 1971; Chang and Haimson, 2000; Haimson and Chang, 2000; Haimson, 2011). Recently, however, true 3-D tests of two sandstones were conducted at \( \sigma \) in excess of both \( \sigma^{\text{brit}} \) and \( \sigma^{\text{ext}} \) (Ingraham et al., 2013a; Ma and Haimson, 2013). These tests confirm and complete the results from the AC and AE tests mentioned above. The results from all these tests summarized and processed in this paper now constitute a solid basis which allows one to draw general conclusions about the dependence of rock behavior on the deviatoric loading configuration (on the Lode angle \( \theta \)). To our knowledge such dependence has not been predicted by micro-mechanical models of geomaterials even qualitatively, and it remains unclear how strongly this dependence will modify the existing models of rock deformation and failure. Based on the presented data we formulate a new three-invariant constitutive model with convex and concave yield surfaces/functions (YFs) which are used for the bifurcation analysis. The results agree with the experimental data for both functions and reveal that the major factors defining the \( \theta \)-dependence of rock response are the \( \theta \)-dependence of the YFs but also of the dilatancy factor which is generally greater for AE than for AC. The theoretical results show that the failure (deformation band) plane can deviate from the intermediate stress direction. The experimental results by Ingraham et al. (2013b) also seem to suggest such a deviation, but more experimental studies are needed to test this prediction.
2. Experimental data and their processing

2.1. Data

The data covering sufficiently large \( \sigma \) range including \( \sigma_{\text{ex}}^{\text{bdt}} \) (\( \sigma_{\text{c}}^{\text{bdt}} < \sigma_{\text{ex}}^{\text{bdt}} \)) are now available for four geomaterials: Solnhofen limestone, rock analog material GRAM1, Castlegate and Bentheim sandstones. As indicated above, there is only data from conventional AC and AE tests for Solnhofen limestone and GRAM1. These two loading configurations correspond to the two extreme values (60° and 0° respectively) of the Lode angle, which is defined as \( \theta = \cos 3J_{3}^{1.5} \), where \( J_{3} \) is the third invariant of stress deviator tensor. All possible deviatoric stress states correspond to the \( \theta \) values in the range 0° to 60°. True 3-D tests cover different \( \theta \) values from this range. The true 3-D data covering \( \sigma_{\text{ex}}^{\text{bdt}} \) are available for two sandstones, Castlegate (Ingraham et al., 2013a) and Bentheim (Ma and Haimson, 2013). The data sets include the three nominal principal stresses \( \sigma_{i} \) (\( i = 1, 2, 3 \)), \( \theta \) (calculated from \( \sigma_{i} \)), and \( \psi \) at failure given in Table 1. The data in this table do not always cover the extreme values of \( \theta \) corresponding to AE and AC. These data were therefore extrapolated for all \( \theta \) using the fact that they are fairly well approximated by a quadratic function as is shown in Fig. 2.

2.2. Constraining the initial yield function

The aim here is to approximate the experimental points \( \sigma_{i} \) as well as \( P^{*} \) or the corresponding points (\( \tau, \sigma, \theta \)) with an analytical function \( \tau(\sigma, \theta) \), having the same form for both Bentheim and Castlegate sandstones (for which the true 3-D data are available). Unfortunately, the \( P^{*} \) values were not measured directly in the papers cited. Therefore we deduce them from \( P^{*} \) measurements in other AC tests. For Bentheim sandstone Baud et al. (2006) have defined \( P^{*} \) to be around 400 MPa. For Castlegate sandstone, \( P^{*} \) has not been defined at all. On the other hand, it is known that the initial yield envelopes for all sandstones are roughly geometrically similar (Wong et al., 1997). Therefore the \( \sigma_{c}/P^{*} \) ratio does not vary much for different materials.

### Table 1

The principal nominal stresses at failure, the calculated \( \theta \) values from these stresses, and measured values of \( \psi \) angle for the two sandstones (from Ma and Haimson, 2013; Ingraham et al., 2013a). The failure stresses correspond to the peaks in the stress–strain curves. When no stress peak occurred (at high \( \sigma \)), the end of the knee in the stress–strain curve was assumed as the failure point for Castlegate sandstone (Ingraham et al., 2013a).

<table>
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<th>( \sigma_{2} ) MPa</th>
<th>( \sigma_{3} ) MPa</th>
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sandstones \((\sigma_c)\) is the mean stress at the crest of yield envelopes). For Bentheim sandstone \(\sigma_c/P_e \approx 0.4\). Assuming the same value of this ratio for Castlegate sandstone, obtain \(P_e = 260\, \text{MPa}\).

The function \(f(\sigma, \theta)\) must meet the following requirements: (i) vanish at \(\sigma = P_e\) for all \(\theta\), (ii) have qualitatively the same trend as the curves in Fig. 1 for the extreme \(\theta\) values, and (iii) approximate the experimental points \(\sigma_i\) not only for these extreme \(\theta\) values, but for all available \(\theta\) values in the range \(0^\circ \leq \theta \leq 60^\circ\). The numerous trials have led to the following simple functional form

\begin{align*}
\sigma (\sigma_1, \sigma_2, \sigma_3) &= a_1 + 2a_2 \sigma_2 + a_3 \sigma_3 + 4a_4 \sigma_2 \sigma_3 + a_5 \sigma_2^2 + a_6 \sigma_3^2.
\end{align*}
\[
\tau(\sigma, \theta) = q_1 + q_2 \sigma + q_3 \sigma^2 + q_4 \sigma^3 + q_5 \sigma^{1.5},
\]

where \( q_i \) are functions of \( \theta \) \((r = 1, 2...5)\). Eq. (1) is plotted along with the approximated experimental points in Fig. 3 for the two sandstones at \( \theta = 0^\circ, \bar{\tau}_{\text{ex}}(\sigma) \), and \( 60^\circ, \bar{\varepsilon}_{\text{ex}}(\sigma) \). The values of \( q_i \) for AC \((\theta = 60^\circ)\) and for AE \((\theta = 0^\circ)\) are given in Table 3.

The obtained functions \( \tau_{\text{AC}}(\sigma) \) and \( \tau_{\text{AE}}(\sigma) \) are then extrapolated for any \( \theta \) as:

\[
\tau(\sigma, \theta) = \begin{cases} 
\tau_{\text{AC}}(\sigma) \times A(\sigma, \theta, w = w_1)/A(\sigma, \theta = 60^\circ, w = w_1) & \text{if } \sigma \leq \sigma_0 \\
\tau_{\text{AE}}(\sigma) \times A(\sigma, \theta, w = w_2)/A(\sigma, \theta = 0^\circ, w = w_2) & \text{if } \sigma > \sigma_0,
\end{cases}
\]

where

\[
A(\sigma, \theta, w) = \left[ 1 + B(\sigma, w) \cos(3\theta) \right]^{w_0}; \quad B(\sigma, w) = \frac{\bar{\sqrt{C(\sigma)}} - 1}{\bar{\sqrt{C(\sigma)}} + 1}; \quad C(\sigma) = \frac{\tau_{\text{AC}}(\sigma)}{\tau_{\text{AE}}(\sigma)},
\]

\( \sigma_0 \) is the mean stress at the intersection of \( \bar{\tau}_{\text{AC}}(\sigma) \) and \( \bar{\tau}_{\text{AE}}(\sigma) \), and \( w \) is the constant which defines the concavity/convexity of the yield surface/function (2); \( w \) can be different for \( \sigma \) smaller \((w = w_1)\) and higher \((w = w_2)\) than \( \sigma_0 \). In other words the YF can be concave (convex) at small and convex (concave) at high \( \sigma \). At \( \sigma = \sigma_0 \) the deviatoric section of the YS is a circle in all cases. In this paper we investigate two cases: \( w = w_1 = w_2 = -0.2 \) and \( w = w_1 = w_2 = -1 \), corresponding respectively to the convex and concave shape of the YS.

Function (2) is reduced to \( \bar{\tau}_{\text{AC}}(\sigma) \) and \( \bar{\tau}_{\text{AE}}(\sigma) \) (shown in Fig. 3) for the extreme \( \theta \) values in all cases. The complete function (2) is plotted in Fig. 4 for both sandstones for the concave model (for the convex model, the surfaces are very similar in this space). It is seen that this function fits the experimental points rather satisfactorily. The correlation coefficient is 0.974 and 0.967 for Bentheim and Castlegate sandstones, respectively. For the convex surface the fit is equally good, which does allow to discriminate between the models.

In Fig. 5 both models (surfaces) are shown in the principal stress space and in the deviatoric plane for Bentheim sandstones.

### Table 3

<table>
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<tr>
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<th>Castlegate</th>
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<td>( q_1 ) (MPa)</td>
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<td>( q_3 ) (MPa)</td>
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<td>( q_5 ) (MPa)</td>
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</table>

**Fig. 4.** Complete (covering the entire \( \theta \) range) yield surfaces from Eq. (2) for Castlegate (a) and Bentheim (b) sandstones in \((\tau, \sigma, \theta)\) space for the concave model. The points are the same as those shown in Fig. 2. The stresses are in megapascals.
sandstone. In this space the concavity/convexity at large and small \( \sigma \) is seen clearly. The concavity is much smaller for Castlegate sandstone (Fig. 6(a)) because the difference between \( \tau_{\sigma \bar{c}} \) and \( \tau_{\sigma \bar{e}} \) is smaller (Fig. 3). The data for Castlegate sandstone (given in Table 1 and obtained by extrapolation in Fig. 2) allow tracing the failure surface in the deviatoric plane for a number of \( \sigma \) values, Fig. 7 (this cannot be done for Bentheim sandstone). This figure shows that both models reflect well the general trend of the geometry change with \( \sigma \). The maximum difference between the two models should be at small \( \sigma \) where the triangularity/concavity of the surface is the highest. It can be noticed that at \( \sigma = 30 \) MPa, the concave model fits the points somewhat better. The correlation coefficient for this model (at this \( \sigma \) value) is 0.92, whereas for the convex model it is 0.83. The same is true for \( \sigma = 60 \) MPa. At larger \( \sigma \), the deviatoric profiles approach a circle in both models (\( \sigma_0 \) for this material is 117 MPa), and then (at \( \sigma = 150 \) MPa) a rounded triangle rotates by 60° with respect to the deviatoric profiles at low \( \sigma \) (Fig. 7) in accordance with Fig. 6. Although the above analysis rather leans in favor of the concave model, the available data does not allow one to make a definitive choice. Therefore we will use both models in the following analysis.

2.3. Dependence of the orientation of the failure plane (of angle \( \psi \)) on \( \sigma \) and \( \theta \)

The \( \psi \) values are generally characterized by a significant dispersion in the experiments, notably at small \( \sigma \) (where rupture
process is sensitive to various sort of heterogeneities including those at the interfaces specimen-steel plattens). This is particularly true in poly 3-D tests (Ingraham et al., 2013a). Nevertheless, the experimental points ($\psi$, $\sigma$, $\theta$) for Bentheim sandstone and GRAM1 can be fitted very satisfactorily by planar surfaces (Fig. 8). The fit is less good for Solnhofen limestone (very old data) and Castlegate sandstone (very large dispersion of the $\psi$ values), Fig. 8. In all cases, however, there is a clear trend of $\psi$ increase with both $\sigma$ and $\theta$. The difference $\Delta \psi$ of $\psi$ values at $\theta$ equal to $0^\circ$ and $60^\circ$ calculated for the same $\sigma$ from the planes approximating the experimental points in Fig. 8 are: $\Delta \psi^B = 12.6^\circ$, $\Delta \psi^C = 6^\circ$, $\Delta \psi^G = 18^\circ$, and $\Delta \psi^S = 11^\circ$ for Bentheim and Castlegate sandstones, GRAM1, and Solnhofen limestone, respectively. One of the objectives of the deformation bifurcation analysis below is to check whether this trend of $\psi$ variation with $\sigma$ and $\theta$ is predicted from the constitutive model with the yield function presented above.

Fig. 6. Deviatoric sections of the yield surface for Castlegate sandstone for the concave (a) and convex (b) models. The sections correspond to the following values of the mean stress: 20 MPa (1); 80 MPa (2); $\sigma = \sigma_0 = 117.5$ MPa (3); 180 MPa (4), and 230 MPa (5). For comparison, $P^* = 260$ MPa.

Fig. 7. Superposition of the deviatoric profiles for the concave (a) and convex (b) models on the experimental points (1–5) for Castlegate sandstone: (1) $\sigma = 30$ MPa; (2) 60 MPa; (3) $\sigma = 90$ MPa; (4) $\sigma = 120$ MPa; and (5) $\sigma = 150$ MPa. For comparison, $\sigma_0 = 117.5$ MPa.
### 3. Constitutive formulation

We assume the yield function (2)

\[
F(\tau, \sigma, \theta) = \begin{cases} 
\bar{F}(\sigma, \theta) & \text{if } \sigma \leq \sigma_0 \\
\bar{F}(\sigma, \theta) & \text{if } \sigma > \sigma_0 
\end{cases}
\]

where \(A(\sigma, \theta)\) is defined in (3) and \(w\) can take two values, \(-0.2\) (convex model) and \(-1\) (concave model). The derivative of this function with respect to the stress tensor \(\sigma_{ij}\) reads

\[
f_{ij} = \frac{\partial F}{\partial \sigma_{ij}} = f_{ij} \sigma + f_{ij} \frac{\partial \sigma}{\partial \sigma_{ij}} + f_{ij} \Omega_{ij},
\]

where \(f_{ij} = \frac{\partial F}{\partial \sigma_{ij}}, f_{ij} = \frac{\partial F}{\partial \sigma_{ij}}, f_{ij} = \frac{\partial F}{\partial \sigma_{ij}}, f_{ij} = \frac{\partial F}{\partial \sigma_{ij}},\)

\[
\Omega_{ij} = \frac{\partial \sigma_{ij}}{\sigma_{ij}} = - \frac{1}{\sqrt{3}} \frac{1}{\sin(3\theta)} \left( \frac{3 s_{ik}s_{kj}}{2} - \sqrt{3} s_{ij} \cos(3\theta) - \delta_{ij} \right),
\]

\(s_{ij}\) is the stress deviator tensor, and \(\delta_{ij}\) is the Kronecker delta. Eq. (5) can be rewritten now as

\[
f_{ij} = \frac{s_{ij}}{2\tau} + \frac{1}{3} f_{ij} \delta_{ij} + f_{ij} \Omega_{ij}.
\]

The flow rule reads

\[
d \varepsilon_{ij} = d \lambda \frac{\partial \Phi}{\partial \sigma_{ij}} = d \lambda \phi_{ij},
\]

where \(\Phi\) is the plastic potential function and \(d \lambda\) is a non-negative scalar function. As in the classical Drucker-Prager model, we assume that \(\phi_{ij}\) has similar to \(f_{ij}\) form

\[
\phi_{ij} = \frac{s_{ij}}{2\tau} + \frac{1}{3} \phi_{ij} \delta_{ij} + \phi_{ij} \Omega_{ij}
\]

Function \(\phi(\sigma, \theta)\) corresponds to the dilatancy factor \(\beta(\sigma, \theta)\) and \(f(\sigma, \theta)\), to the internal friction coefficient \(\alpha(\sigma, \theta)\). By processing experimental data sets from AC tests on three geomaterials, it was shown (Mas and Chemenda, 2015) that for
this loading configuration $\beta(\sigma) = \alpha(\sigma) - \alpha_0$, where $\alpha_0$ varies slowly with $\sigma$ and the inelastic strain, and therefore can be considered constant to a first approximation. Here we assume that the same relation holds for any $\theta$:

$$\beta(\sigma, \theta) = \alpha(\sigma, \theta) - \alpha_0,$$  \hspace{1cm} (10)

Therefore

$$g_{ij} = f_{ij} - \frac{1}{3}\alpha_0 \delta_{ij}.$$  \hspace{1cm} (11)

Parameter $\alpha_0$ varies for the materials analyzed in (Mas and Chemenda, 2015) from 0 to 0.3. We assume intermediate values of $\alpha_0$ for Bentheim ($\alpha_0 = 0.15$) and Castlegate ($\alpha_0 = 0.2$) sandstones analyzed in Section 5 (the explanation of this choice will be given below).

In the elastic-plastic domain, the increments of stress and strain are related as

$$d\sigma_{ij} = L_{ijkl} d\varepsilon_{kl}.$$  \hspace{1cm} (12)

The elasto-plastic stiffness tensor $L_{ijkl}$ is the sum of elastic $L_{ijkl}^{el}$ and plastic $L_{ijkl}^{pl}$ stiffness tensors

$$L_{ijkl} = L_{ijkl}^{el} + L_{ijkl}^{pl},$$  \hspace{1cm} (13)

where

$$L_{ijkl}^{pl} = \left( K - \frac{2}{3}G \right) \delta_{ij} \delta_{kl} + G (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$  \hspace{1cm} (14)

$K$ and $G$ are the bulk and shear moduli, respectively.

The total incremental strain $d\varepsilon_{ij}$ is decomposed on the increments of the elastic $d\varepsilon_{ij}^{el}$ and inelastic $d\varepsilon_{ij}^{pl}$ strains:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{el} + d\varepsilon_{ij}^{pl},$$  \hspace{1cm} (15)

where $d\varepsilon_{ij}^{el}$ is related to stresses by Hook’s equations

$$d\sigma_{ij} = L_{ijkl}^{el} d\varepsilon_{kl}^{el},$$  \hspace{1cm} (16)

and inelastic strains are obtained from the flow rule (8).

Combining (8), (15) and (16) obtain

$$d\sigma_{ij} = L_{ijkl}^{el} (d\varepsilon_{kl} - d\varepsilon_{kl}),$$  \hspace{1cm} (17)

Considering that

$$d\varepsilon_{ij}^{pl} = (2d\varepsilon_{ij}^{p})^{1/2},$$  \hspace{1cm} (18)

and

$$d\varepsilon_{ij}^{p} = d\varepsilon_{ij}^{pl} - \delta_{ij} d\varepsilon_{kk}^{p}/3,$$  \hspace{1cm} (19)

obtain from (8)

$$d\varepsilon_{ij}^{pl} = d\lambda \sqrt{2\left(g_{ij} g_{ij} - \frac{1}{3} g_{kk}^2 \right)},$$  \hspace{1cm} (20)

Substituting (7), (11), (17) and (20) into the consistency equation

$$d\sigma = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon_{ij}^{pl}} d\varepsilon_{ij}^{pl} = 0,$$  \hspace{1cm} (21)

yields

$$l_{kl} d\varepsilon_{kl} - d\lambda \omega = 0,$$  \hspace{1cm} (22)

where

$$\omega = l_{kl} g_{kl} - H \sqrt{2\left(g_{ij} g_{ij} - \frac{1}{3} g_{kk}^2 \right)},$$  \hspace{1cm} (23)

$l_{kl}$ and $H = -\partial F/\partial \varepsilon_{ij}^{pl}$ is the hardening modulus.

Solving (22) for $d\lambda$ and substituting the result into (17), obtain (considering (13)):
\[ L_{ijkl} = L_{ijkl}^0 - \frac{1}{\alpha} t_{ij}^0 h_{kl} \]  
\[ L_{ijkl}^p = - \frac{1}{\alpha} t_{ij}^p h_{kl} \]  
where \( t_{ij}^0 = L_{ijkl}^0 g_{kl} \).

Considering (7), (11), (14) and that \( \Omega_i \Omega_j = r^{-2} \), \( L_i \), \( L^0_i \), and \( \omega \) can be rewritten as

\[ L_i = \frac{3 q_i}{r} G + f_{ij} K \delta_{ij} + 2 f_{ij} G \Omega_i \]  \[ t_{ij} = \frac{3 q_i}{r} G + (f_{ij} - a_0) K \delta_{ij} + 2 f_{ij} G \Omega_i \]  \[ \omega = G \Lambda + H \sqrt{\Lambda} + f_{ij} (f_{ij} - a_0) K, \]  
where \( \Lambda = 1 + (f_{ij} / r^2) \). The dilatancy factor is defined as the ratio \( d \varepsilon_p / d \sigma_p \) where \( d \varepsilon_p = d \lambda g_{ij} \) and \( d \sigma_p = d \sqrt{\Lambda} \). Therefore

\[ \beta(\sigma, \theta) = g_{ij}(\sigma, \theta) / \sqrt{\Lambda(\sigma, \theta)} \]  

4. Deformation bifurcation condition and orientation of deformation bands

A deformation bifurcation condition (condition of a constitutive instability and of the onset of deformation localization) is found from the equation relating directional cosines \( n_i \) of the localization band and \( L_{ijkl} \) (Hill, 1962; Mandel, 1966; Rudnicki and Rice, 1975) or more complex tensor when considering the formation of the conjugate band network (Garagash, 1981; Chemenda, 2007). The approaches in solving this equation can be somewhat different (Rudnicki and Rice, 1975; Molenkamp, 1985; Vermeer, 1990; Perrin and Leblond, 1993; Vardoulakis and Sulem, 1995), but they lead to the same result, the expressions for the normalized critical hardening modulus \( h = H / G \) at the onset of bifurcation, and for the angle \( \psi \) between the forming bands and \( \sigma_1 \).

The continuous bifurcation problem can be reduced to the analysis of the quadratic equation (e.g., Chemenda, 2007)

\[ \xi^2 + \xi Q(L_{ijkl}) + R(L_{ijkl}) = 0, \]  \[ Q = \frac{2 L_{ijkl} (L_{ijkl} - L_{3333})}{L_{1111} L_{3333} - L_{1111} L_{3333} + L_{1111} L_{3333} + L_{1111} L_{3333} + L_{1111} L_{3333} + L_{1111} L_{3333}} \]  \[ R = \frac{L_{1111} L_{3333} - L_{1111} L_{3333} - L_{1111} L_{3333} + L_{1111} L_{3333} + L_{1111} L_{3333} - L_{1111} L_{3333}}{L_{1111} L_{3333} - L_{1111} L_{3333} - L_{1111} L_{3333} + L_{1111} L_{3333} + L_{1111} L_{3333} + L_{1111} L_{3333}} \]  
and

\[ \xi = n_1^2 - n_2^2 = \cos 2 \psi \]  
(band normal \( n_i \) is defined in the principal stress space; it is assumed as usual that deformation bands are parallel to the intermediate principal stress, i.e., \( n_2 = 0 \)). The deformation bifurcation is possible when (30) has real roots, i.e., when \( Q^2 - 4R \) vanishes. The maximum (critical) hardening modulus \( h_c \) at which the bifurcation is possible is found therefore from the equation

\[ Q^2 - 4R = 0, \]  \[ \xi = -\frac{Q}{2}. \]  

For the above constitutive model, functions \( Q \) and \( R \) are reduced to the following expressions

\[ Q = 2(2\nu - 1) \frac{l_1 t_{ij}^0 - l_3 t_{ij}^0}{(l_1 - l_3)(l_1^0 - l_3^0)} \]  
and
\[ R = \frac{8G\omega(1 - \nu) + l_1 l_4^* + l_2 l_4 l_1 + (4\nu - 3)(l_1 l_4^* + l_2 l_4^*)}{(l_1 - l_2)(l_1^* - l_2^*)}. \]  

(36)

\( l_1 \) and \( l_4^* \) are from the expressions (26) and (27) written in the principal stress space where

\[ \Omega_1 = -\frac{\sin(\theta)}{\sqrt{3} \tau}; \quad \Omega_2 = \frac{\sin(\theta) + \sqrt{3} \cos(\theta)}{2 \sqrt{3} \tau}; \quad \Omega_3 = \frac{\sin(\theta) - \sqrt{3} \cos(\theta)}{2 \sqrt{3} \tau}; \]

(37)

\( \nu \) is the Poisson's ratio. The expression for the critical hardening modulus is found then from (33)

\[ h_{cr} = \frac{Q^2(l_1 - l_2)(l_1^* - l_2^*) + 4(3 - 4\nu)(l_1 l_4^* + l_2 l_4^*) - 4(l_1 l_4^* + l_2 l_4^*) - 32 G(1 - \nu)(G\lambda + K_f g_f)}{32 G^2(1 - \nu)\sqrt{\lambda}}. \]

(38)

5. Application of the theoretical results to the true 3-D experimental data

5.1. Dependence of the orientation of deformation bands (faults) on \( \sigma \) and \( \theta \)

Fig. 9 shows plots \( \psi(\sigma, \theta) \) (red surfaces) from ((32), (34) and (35)) superposed on the experimental points and the approximating planes from Fig. 8 for both sandstones and the concave model (for the convex model the result is almost the same because in the considered \( \sigma \) range both models are very close). It is seen that in spite of the data scatter and the model simplifications, the fit between the surfaces is rather good for both materials. At least the theoretically predicted trend of the \( \psi \) variation with \( \sigma \) and \( \theta \) is the same as defined by the experimental data. Since the ‘experimental’ and theoretical surfaces are close to parallel, the theoretically predicted difference of \( \psi \) values at the extreme \( \theta \) values and constant \( \sigma \) should be close to those defined above from the experimental data for the two materials: \( \Delta\psi^B = 12.6^\circ, \Delta\psi^C = 6^\circ \). Note that these values are very different for the different materials and the model captures this difference.

It is generally assumed that the deformation localization plane is parallel to the intermediate principal direction or to \( \sigma_2 \), which follows from both theoretical and experimental works. In terms of the deformation bifurcation condition, it means that the critical hardening modulus \( h_{cr}(l_1, l_2, l_4^*, l_4^*) \) as defined in (38) is always larger than \( h_{cr}(l_2, l_1, l_4^*, l_4^*) \) (subscript "1" is replaced here by "2") or \( h_{cr}(l_1, l_2, l_4^*, l_4^*) \) where subscript "3" is replaced by "2" in (38). It is not surprising therefore that the hardening modulus for the case when the deformation localization plane is parallel to the maximum compression stress \( \sigma_1 \)

\[ h_{cr}^B(\sigma, \theta) = h_{cr}(l_1, l_2, l_4^*, l_4^*) \]

(39)

is very negative in Figs. 10(a) and (d) and 11(a) and (d) (except at large \( \sigma \)) for both materials and both models \( (h_{cr}(l_1, l_2, l_4^*, l_4^*) \) is still more negative). The hardening modulus \( h_{cr}^B(\sigma, \theta) = h_{cr}(l_1, l_2, l_4^*, l_4^*) \) defining the deformation bifurcation along \( \sigma_2 \)-parallel plane has large negative values only at large \( \sigma \) and small \( \theta \), and at small \( \sigma \) and large \( \theta \) (Figs. 10(b) and (e) and 11(b) and (e)) where deformation localization is probably impossible as very high softening is required (the localization was not experimentally evidenced in these domains). It follows that the difference \( \Delta h_{cr} = h_{cr}^B - h_{cr}^C \) is positive in most of the \((\sigma,\theta)\) range except in the domain of large \( \sigma \) (Figs. 12(a) and (b), 13(a) and (b) and 14) where deformation bands should be therefore parallel to \( \sigma_1 \) and not to \( \sigma_2 \). In this domain, the angle \( \psi^* \) between the band and \( \sigma_2 \) direction is defined,
considering ((32), (34), and (35)), from
\(\psi^* = -\frac{1}{Q} I_2, I_3, I_4.\)

For the concave model \(\Delta h_{cr}^c\) is negative at large \(\theta\) almost in the whole range (Figs. 12(a), 13(a)), whereas for the convex model (Fig. 14), this is the case only in the range of small \(\theta\) where Eq. (40) does not have real solution for \(\psi^*\). The deformation localization along the \(\sigma_1\)-parallel plane is therefore impossible for the convex model (it does not mean that for this model the localization is impossible along a plan oriented between \(\sigma_1\) and \(\sigma_2\) directions). It is only possible for the concave model at large \(\theta\) where \(\Delta h_{cr} \leq 0\) and \(\psi^*\) is real (Figs. 12, 13). It is seen in these figures that the difference between \(\psi^*\) and \(\psi\) is small for Castlegate sandstone (Fig. 12(c)) and large for Bentheim sandstone (Fig. 13(c)). This is because the concavity for the Castlegate sandstone is smaller than for Bentheim sandstone (Figs. 5 and 6).

To appreciate the impact of the third invariant of the stress tensor on the mechanical response, we present in Figs. 10(c) and 11(c) the plots of \(h_{cr}^d\) obtained from (38) by setting \(\tau_\sigma \sigma = \tau_\sigma \sigma^c\), which reduces this expression to the formula

\[
h_{cr}^d = \frac{1 + \nu}{9(1 - \nu)} (f_\sigma - g_\sigma)^2 - \frac{1 + \nu}{\sqrt{3}} \left[ \cos \left( \frac{2\pi}{3} - \theta \right) + \frac{f_\sigma + g_\sigma}{2\sqrt{3}} \right]^2.
\]

(41)

\(\Delta h_{cr}^d = h_{cr}^d - h_{cr}^c\), where \(h_{cr}^c\) is for the concave model. The grey semitransparent planes show a zero-level of the hardening moduli. The stresses are in megapascals.

Fig. 10. Critical hardening moduli versus \(\sigma\) and \(\theta\) for Castlegate sandstone: (a) and (d) \(h_{cr}^*\) from (39); (b) and (e) \(h_{cr}^c\) from (38); (c) \(h_{cr}^d\) from (41); (f) \(\Delta h_{cr}^d = h_{cr}^d - h_{cr}^c\), where \(h_{cr}^c\) is for the concave model. The grey semitransparent planes show a zero-level of the hardening moduli. The stresses are in megapascals.

functions \(f_\sigma\) and \(g_\sigma\) depend here only on \(\sigma\) and correspond respectively to the internal friction coefficient and the dilatancy factor. It is seen that for both materials \(h_{cr}^c(\sigma, \theta)\) for both models (Figs. 10(b) and (e) and 11(b) and (e)) and \(h_{cr}^d(\sigma, \theta)\) (Figs. 10(c), and 11(c)) are very different from each other as well. In particular, the \(\theta\) range with positive or slightly negative hardening modulus is much wider for \(h_{cr}^c(\sigma, \theta)\) than for \(h_{cr}^d(\sigma, \theta)\). The plots of \(\Delta h_{cr}^d = h_{cr}^d - h_{cr}^c\) in Figs. 10(f) and 11(f) show that \(h_{cr}^c\) is generally larger than \(h_{cr}^d\) meaning that the \(\theta\)-dependence of the properties encourages the deformation localization.
5.2. Brittle-ductile transition

This transition occurs at a sufficiently high pressure (mean stress) when the stress–strain curves do not show a stress reduction (or only a small and smooth one) with deformation (Wong and Baud, 2012), i.e., when the hardening modulus is not reduced to negative values with deformation \( \gamma_p \) but remains zero or slightly positive. This can occur when the material cohesion or the internal friction coefficient or both increase with \( \gamma_p \). On the other hand, the available experimental information suggests that the brittle-ductile transition is related to the transition from dilatant to compactive deformation regimes (Wong et al., 1997). The compaction is known to increase the material cohesion starting from a certain \( \sigma \) value, while dilatancy is accompanied by material decohesion. That is why the compactive deformation localization bands (pure compactive/compaction or shear-compactive) do not typically evolve to fractures (displacement discontinuities), while dilatant bands (pure dilatant/dilatancy or shear-dilatant) do, not only in laboratory experiments (Wong et al., 1997; Bésuelle et al., 2000; Fortin et al., 2006; Nguyen et al., 2011; Chemenda et al., 2011) but also in nature (Fig. 15). The bands with compactive deformation component appear in the field as highs (more cohesive material within the band is altered and eroded more slowly than the host material), Fig. 15(a), while bands with dilatant component represent troughs, Fig. 15(b) and (c).

When the cohesion within the evolving compactive band becomes sufficiently high, the band widens and/or new bands grow in other places (in the laboratory tests (Mair et al., 2000) as well as field and numerical models (Chemenda et al., 2012)), resulting in a plateau in the stress–strain–curves and a band network observed in numerous tests at brittle-ductile transition (Handin et al., 1967; Wong et al., 1997; Bésuelle et al., 2000; Fortin et al., 2006; Tembe et al., 2008; Nguyen et al., 2011). In a dilatant regime, one band/fracture is typically formed in the specimen, which is accompanied by the stress drop. Therefore, the brittle-ductile transition occurs when the dilatancy factor \( \beta \) (which depends on both \( \sigma \) and \( \theta \) (Fig. 16) and evolves with the inelastic strain (Mas and Chemenda, 2015)) is around zero. The curves \( \sigma(\theta) \) as well as \( \psi(\theta) \) (from (29), (32), (34), (35)) for the two close to zero \( \beta \) values, \( \beta = \pm 0.05 \), are plotted for the two materials in Fig. 17. It is seen that \( \sigma \) at brittle-ductile transition (at \( \beta \approx 0 \)) is much higher for \( \theta = 0^\circ \) than for \( \theta = 60^\circ \) (Figs. 17(a) and (c)), in accordance with the experimental data.

It is more difficult to make a quantitative comparison. The experimental data only allow very approximate estimations of

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![Fig. 11. Critical hardening moduli versus \( \sigma \) and \( \theta \) for Bentheim sandstone: (a) and (d) \( h_{cr}^c \) from (39); (b) and (e) \( h_{cr} \) from (38); (c) \( h_{cr}^dp \) from (41); (f) \( \Delta h_{cr}^dp = h_{cr} - h_{cr}^dp \), where \( h_{cr} \) is for the concave model. The grey semitransparent planes show a zero-level of the hardening moduli. The stresses are in megapascals.](image-url)
\[ \sigma_{cbdt} \text{ and } \sigma_{exbdt} \]. For Bentheim sandstone they are roughly \( \sigma_{cbdt} = 150 \text{ MPa} \) and \( \sigma_{exbdt} = 210 \text{ MPa} \) (Ma and Haimson, 2013), and for Castlegate sandstone, \( \sigma_{cbdt} = 70 \text{ MPa} \) and \( \sigma_{exbdt} = 120 \text{ MPa} \) (Ingraham et al., 2013a). The definition of the \( \psi \) values at the transition is still less certain. Very roughly, they are: \( \psi_{cbdt} = 45^\circ \) and \( \psi_{exbdt} = 38^\circ \) for Bentheim sandstone, and \( \psi_{cbdt} = 45^\circ \) and \( \psi_{exbdt} = 40^\circ \) for Castlegate sandstone. Fig. 17 shows that the theoretical predictions are generally coherent with these values in spite of the simplifying assumptions in the model such as, for example, assuming \( \alpha_0 \) to be constant over the whole range of

Fig. 12. Plots \( \Delta h_{cr}(\theta) = h_{cr}(\theta) - h_0^2(\theta, \theta) \) (a); \( \Delta h_{cr}(\theta) \) (b) and both \( \psi(\sigma) \) and \( \psi^*(\sigma) \) at \( \theta = 46^\circ \) (c) for Castlegate sandstone and the concave model. The grey semitransparent plane in (a) corresponds to \( \Delta h_{cr} = 0 \). The stresses are in megapascals.

Fig. 13. Plots \( \Delta h_{cr}(\theta) = h_{cr}(\theta) - h_0^2(\theta, \theta) \) (a); \( \Delta h_{cr}(\theta) \) (b) and both \( \psi(\sigma) \) and \( \psi^*(\sigma) \) at \( \theta = 40^\circ \) (c) for Bentheim sandstone and the concave model. The grey semitransparent plane in (a) corresponds to \( \Delta h_{cr} = 0 \). The stresses are in megapascals.
The values of this parameter for Bentheim ($\alpha_0 = 0.15$) and Castlegate ($\alpha_0 = 0.2$) sandstones were chosen within the range of the available estimations (0–0.3) to obtain the position of $\sigma(\theta)$ and $\psi(\theta)$ curves as shown in Fig. 17. The fit between the theoretical curves and experimental data in this figure can be further improved taking into account that $\alpha_0$ is not constant but decreases slowly with $\sigma$ (Mas and Chemenda, 2015). However our aim here is not to obtain an exact fit of the model to the data, but to see whether the model integrating the $\theta$-dependence of the properties in the simplest way captures the main features of the rock behavior that are not captured by the 2-invariant models.

5.3. Convex versus concave shape of the yield surface and the orientation of the deformation localization bands

The Drucker postulate (Drucker, 1959) requires the convex shape of the yield surface (YS). Most experimental data (available mostly for small $\sigma$ values compared to $P^*$) also suggest a rather convex shape of the YSs. Therefore practically all
3-invariant constitutive models proposed previously are convex (e.g., Willam and Warnke, 1975; Van Eekelen, 1980; Hsieh et al., 1982; Desai and Salami, 1987; Lade and Kim, 1988). On the other hand, the Drucker postulate was shown by Mróz (1963), Mandel (1966) and others to be a sufficient but not necessary condition of stability and uniqueness in dynamic and static problems for nonassociated plasticity models that are typical for geomaterials. There are also experimental data indicating the concavity (Kirkgard and Lade, 1993; Lade, 2002). The lack of such data can be explained simply by the fact that the concavity is expected only at very small \( \sigma \) and in this paper also at very high \( \sigma \), Fig. 5. Most of the limited available true 3-D data are for the brittle/dilatant deformation domain \( (\sigma < \sigma^{bdt}) \) and not very small \( \sigma \). Some theoretical and/or numerical studies within the 3-invariant constitutive framework (Borja, 2004; Challa and Issen, 2007) use only convex models, but others (Collins, 2003) admit the concavity of YSs.

Our data analysis rather suggests the concavity at small \( \sigma \) (Fig. 7) but does not prove it, given the data scatter and limited volume. The data does not exist at all for large \( \sigma \) (Fig. 4) in the ductile/compactive domain, where the concavity is also suggested (Figs. 5 and 6). Most results of the theoretical analysis in this paper are not very sensitive to the convexity/concavity of YSs except the prediction of the orientation of deformation bands. They can be parallel to \( \sigma_1 \) (and not to \( \sigma_2 \)) at very high \( \sigma \) for the concave YS. This suggests that the deviation of the band orientation from the \( \sigma_2 \) direction can occur at smaller \( \sigma \) for the concave and possibly also for the convex model. The experimental evidences of a parallelism of deformation bands to the \( \sigma_1 \) direction could be an indirect proof of the YS concavity. The true 3-D experimental studies seem on the contrary to indicate a parallelism of the deformation bands/faults to \( \sigma_2 \) (e.g., Desrues et al., 1985; Haimson, Chang, 2000), but these studies are limited to \( \sigma < \sigma^{bdt} \). To our knowledge there is only one work (Ingraham et al., 2013b) specifically oriented towards the investigation of the band orientation including in a compactive deformation regime. These authors used acoustic emissions to detect the deformation localization bands and define their orientation. This technique is the most suitable for identifying the compactive bands (shear-compactive or pure compactive) forming at high \( \sigma \) (where the concavity is expected) in the experiments as this band type is difficult to identify visually on the postmortem specimens (in the outcrops, they are clearly seen due to the differential erosion, Fig. 15(a)). The results of Ingraham et al. (2013b) suggest that the angle between the band plane and \( \sigma_2 \) may not be zero even at small \( \sigma \) and may increase with the \( \sigma \) growth. The definition of the band orientation at large \( \sigma \) is hampered however by a number of uncertainties, which does not allow drawing definitive conclusions. More experimental studies are needed.
6. Conclusion

The dependence of the mechanical response of geomaterials on the Lode angle $\theta$ has long been known but only for small mean stresses $\sigma$ corresponding to the brittle deformation regime, since most of the experimental information was limited only to such $\sigma$ (except for the only $\theta$ value, $\theta=60^\circ$, corresponding to axisymmetric compression). Nor did micro-mechanical studies bring insights into the possible mechanism of this dependence that would help in the formulation of 3-invariant constitutive models for a wide range of $\sigma$. Although still limited, the data from true 3-D as well as both compression and extension axisymmetric tests available now for four geomaterials allow to draw some general conclusions and formulate a first-order constitutive model for deformation regimes from brittle faulting to ductile flow. The most striking property revealed by these data is a strong $\theta$-dependence of the mean stress $\sigma_{\text{bdt}}$ at brittle-ductile transition. The ratio $\sigma_{\text{bdt}}/\sigma_{\text{bdt}}$ varies from 1.4 to 1.7 for the four materials studied, which agrees with the model predictions. This difference was shown to be due to the $\theta$-dependence of the yield function as well as of the dilatancy factor $\beta$ which is generally larger for $\theta=0^\circ$ (AE) than for $\theta=60^\circ$ (AC), Fig. 16. Such dependence of $\beta$ on $\theta$ was not postulated but obtained in the model. This dependence is also confirmed by the processing of the experimental data for GRAM1 material (for $\theta=60^\circ$ the results have been reported in [34]) and for $\theta=0^\circ$, the paper is in preparation). For other geomaterials, only isolated $\beta$ values and only for $\theta=60^\circ$ are available (it should be noted that the correct definition of $\beta$ represents a very delicate procedure, which requires a separation of elastic and inelastic strains using a large and good quality experimental data set). It follows from the above that the inelastic straining, strain localization and failure of non-uniformly stressed materials/structures should be strongly $\theta$-dependent. Theoretical and numerical modeling of these processes therefore requires 3-invariant constitutive models.

The proposed yield function/surface (YS) can be either convex or concave at small and large $\sigma$ (at intermediate $\sigma$ values, the surface is always convex), which depends on the value of one parameter. Most results of the theoretical analysis are not
very sensitive to the convexity/concavity of YS except the prediction of the orientation of deformation bands. They can be parallel to \(\sigma_1\) (and not to \(\sigma_2\)) at very high \(\sigma\) for the concave YS. This suggests that the deviation of the band orientation from the \(\sigma_2\) direction can occur at smaller \(\sigma\) for the concave and possibly also for the convex model. Specific experimental studies are needed to test this prediction.

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References


